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Dynamics of opinion formation in hierarchical social networks: Network structure and initial bias

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Abstract. The dynamics of opinion formation based on a majority rule model is studied in a network with the social hierarchical structure as one of its limits. The exit probability is found to change sensitively with the number of nodes in the system, but not with the parameter of homophyly characterizing the network structure. The consensus time is found to be a result of non-trivial interplay between the network structure characterized by the parameter of homophyly and the initial bias in opinion. For unbiased initial opinion, a common consensus is easier to be reached in a random network than a highly structured hierarchical network and it follows the behavior of the length of shortest paths. For biased initial opinion, a common consensus is easier to be reached in a hierarchical network, as the local majority opinion of the groups may take on the biased opinions and hence be the same.

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1 Introduction

The physics of networks has attracted much interest in recent years not only among physicists, but also among scientists in other disciplines [1–3]. The structures of networks, dynamical processes on networks, and the interplay between the structural and the dynamical properties are the important questions. Structurally, it was found that many real-world networks exhibit the small-world phenomena [4]. Various dynamical processes have been studied on small-world structures [5–7]. The physics of dynamical processes on networks is particularly rich. In these processes, the nodes in a network may be taken to be individuals and the links may be taken to be paths through which information flows. These dynamical processes may be epidemics in a population, cultural assimilation, opinion formation, mailing, voting, or decision making in a competing environment for limited resources. A good review on dynamical processes in complex networks has been given recently by Bocaletti et al. [8]. Here, we study the dynamics of opinion formation in a network that interpolates random networks and social hierarchical networks.

The statistical physics of social dynamics has become an active area of research [9]. Among the interesting problems is that of opinion formation. One of the prototypical models of opinion formation is based on the simple majority rule [10]. The nodes in a system can take on two possible states, say $+1$ and -1. The majority rule amounts to an updating scheme in which a node and its connected neighbors take on the state of the local majority. Slightly rephrased, the model is closely related to magnetic models in statistical physics. The probability of reaching a consensus, e.g., $+1$, for a given initial bias in opinion and the time to reach a consensus have been studied in the mean field limit and regular lattices [11,12]. Recently, we have studied the model in a network in which a number of shortcuts between nodes are randomly added into an underlying regular lattice [13]. Newman and Watts showed that such a network can be tuned from a regular lattice to a small-world [14], and the length of shortest paths changes from a linear dependence on the network size to a logarithmic dependence when a small fraction of shortcuts are added. The dynamics of opinion formation was found to depend sensitively on the length of shortest paths in this Newman-Watts network model. Besides the majority rule, another model of opinion formation is the voter model, which has been implemented on lattices [15–17] and on complex networks [18]. In this model, a node is updated by taking on the opinion of a randomly selected neighbor [19]. A key difference is that the voter model considers only pairwise interactions, while the majority rule considers the interactions among the nodes in a group.

To understand the dynamics of opinion formation better, one needs a more appropriate network that shows the characteristics of social networks. Besides having

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small length of shortest paths and high clustering coefficient, both typical of small-world behavior, an important character of social networks is its "searchability". Kleinberg [20,21] pointed out that the well-known small-world experiments of Milgram [22–24] demonstrated not only that the existence of short paths between randomly chosen individuals in a population, but also the property that individuals could find each other through such paths by using only local information about the network. Thus, in addition to being small-world, social networks are also searchable [25]. Kleinberg [21] also proved that while short paths do exist in networks constructed by rewiring or adding links to an underlying regular lattice [5,14], the nodes cannot find each other readily using methods of local search. Motivated by the observation that individuals in a population are usually classified into hierarchical structures, Watts et al. [26] proposed a network construction that interpolates random networks and social hierarchical networks that are searchable. The construction is, obviously, more appropriate for studying the dynamics of opinion formation.

In the present work, we study the dynamics of achieving a uniform opinion in hierarchical social networks. In Section 2, the construction of the hierarchical network and the opinion formation model based on the majority rule are introduced. The behavior of the averaged length of shortest paths as a function of the parameter of homophyly is discussed, as it is an important parameter in discussing the opinion formation results that follow. In Section 3, results on how the exit probability and the consensus time depend on the structure of the network and initial bias in opinion are presented. The results are discussed within the context of the interplay between the network structure and the initial bias. A summary is given in Section 4.

2 Hierarchical networks and opinion formation model

We study opinion formation in a model of social networks. First, we discuss the network construction proposed by Watts et al. Motivated by the observation that entities in a society are generally grouped in a hierarchical fashion, Watts et al. [26] proposed a model of constructing a hierarchical social networks. Figure 1 shows schematically a hierarchical structure in partitioning a population. It has been argued that many social systems of N individuals are naturally divided into a layered structure. The top layer represents the whole population of N individuals. In a hierarchy, these individuals are then divided into b groups. Each group is further divided into b subgroups, and the process of division is continued until each individual belongs to a group with some functional group size g. After $L-1$ divisions, the structure has L layers. The total number of individuals is given by $N = \langle g \rangle b^{L-1}$, where $\langle g \rangle$ is the mean group size of the lowest layer. Typically, $\langle g \rangle$ is of the order of unity or ten, as evidence from the typical size of a section of specific functionality, e.g., auditing, marketing, and accounting, in a company, and the typical size of research groups in academic institutions.

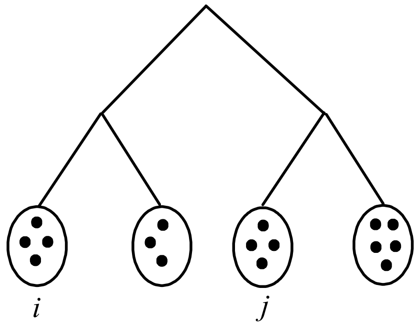

Fig. 1. Individuals in a population can typically be classified into several hierarchies. The grouping of individuals in one of these hierarchies is shown schematically. The hierarchy has a branching ratio $b = 2$, total number of layers $L = 3$, and mean group size $\langle g \rangle = 4$ at the lowest layer. Two individuals have different social distances in different hierarchies.

The notion of social distance [26] plays a key role in studying problems such as epidemics, spread of rumors, and opinion formation. A person is more likely to affect a close friend or to be influenced by a close friend. The social distance provides a measure of this "closeness". In a hierarchy, social distance can be conveniently defined. The social distance x_{ij} between two individuals i and j in a hierarchy is defined as the number of levels to reach their common ancestor in the hierarchy, with the lowest layer set to be level 1. Practically, it means to search upwards in the hierarchy until i and j meet. The maximum social distance between two individuals is thus L and the distance between any pair of individuals belonging to the same group is 1. Take the hierarchy in Figure 1 as an example, we have $x_{ij} = 3$. The social distance thus measures the similarity between individuals, and the probability of acquaintance between individuals i and j increases with decreasing social distance.

Another typical feature of social networks is that a population can be characterized by a number H of hierarchies [26]. A hierarchical structure can be constructed by considering one attribute of identity, e.g., geographical location, types of employment, religious beliefs, and research interests. For $H > 1$, a node i can be close to a node j in one hierarchy and to another node k in another hierarchy, while j and k may be far apart in both hierarchies, i.e., nodes i and j have different social distances in different hierarchies. An example among physicists is that you know your colleague in the office next door and your collaborator in another continent well, but they may not know each other. This feature helps in the connections between nodes in the system.

Watts et al. [26] introduced a construction that gives the hierarchical structure of social networks as one of its limits. The algorithm goes as follows. Consider a system with N nodes that can be characterized by H hierarchies. Before any links are established, the N nodes are distributed randomly into the groups in the lowest layers in each of the H hierarchies, for given values of $\langle g \rangle$, b, and L.

The N nodes are, in general, distributed in different ways into the lowest layer of each hierarchy. Note that this initial configuration is *not* the resultant network. It simply serves to define the social distance x_{ij} between two nodes in each of the hierarchies. The links connecting the nodes are established by [26]: (i) randomly pick a hierarchy; (ii) randomly pick a node i; (iii) decide on the length of the link to be formed probabilistically, using the probability $P(x) = C\exp(-ax)$ for a link with another node that is a social distance x ($x = 1, \ldots, L$) away from node i, with C being a normalization constant and a being a tunable parameter; (iv) collect all the nodes that are of the desired distance from node i in the chosen hierarchy and randomly pick one node j to establish a link; (v) repeat the process until a desired value of mean degree $\langle k \rangle$ is achieved. Typically, we choose $\langle k \rangle = \langle g \rangle - 1$. Here, the parameter a is a parameter of homophyly [26]. For one hierarchy $H = 1$, it is easy to see that short links are favored for $a \gg 1$ and hence the resultant network is of the structure shown in Figure 1. For $a = -\ln b$, any two nodes have equal chance to be linked and a random network results. In the presence of several hierarchies, a key feature is that while nodes i and j may have a very long social distance in one hierarchy, they could reach each other easily if i and k (and j and k) have short social distance in the other hierarchies. This multi-hierarchy nature of social networks has been shown to be important in problems such as searchability [26] and epidemics [27].

Here, we study an opinion formation dynamics based on a majority rule [10–12] on the social network. Each node can be in one of two possible states represented by $+1$ and -1. Initially, a fraction p of nodes take on $+1$ and $(1-p)$ take on -1. These states, for example, represent two opposite opinions. Following the model studied by Redner and co-workers [11,12], the states of the nodes evolve in time according to the following updating rules. At each time step, one node is chosen randomly. The chosen node and his connected neighbors are then considered collectively for updating. All the nodes in the cluster of nodes will then be updated to take on the state of the local majority. The updating rule thus represents a consensus within the cluster by taking the majority opinion. The procedure is then repeated until all the nodes take on the same state, i.e., when a final state of consensus is reached in the whole system. Obviously, one could rephrase the problem and the dynamics in terms of up $(+1)$ and down (-1) spins as in magnetic systems. This model of opinion formation has previously been studied on regular lattices and lattices that exhibit small-world properties [13]. The social network studied here has the advantage of reflecting the layer-by-layer grouping in many real-life networks involving people, and thus it is uniquely suitable for studying the dynamics of opinion formation. The interesting question is how the structure of the network affects the exit probability and consensus time.

For $H = 1$ and large values of a, the network consists of locally connected groups that are isolated from each other and thus the network is not connected. For $H > 1$, there are a few links that connect the groups even for large a.

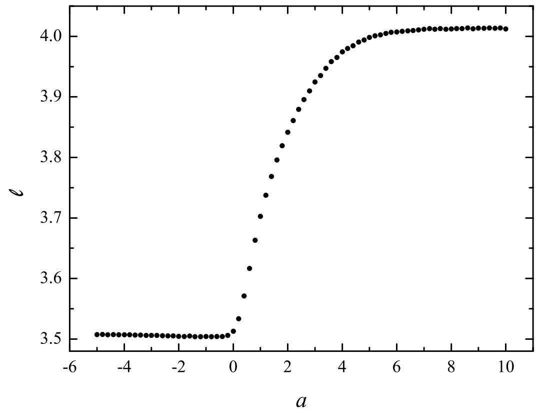

Fig. 2. The averaged length of shortest paths ℓ as a function of a in hierarchical networks with $H = 2$, $L = 8$, $b = 2$, $\langle g \rangle = 10$ and $\langle k \rangle = 9$. For each value of a, the result is obtained by averaging over 100 different realizations of the network.

We have checked numerically that for $H = 2$ and for a large range of a, the resulting network is connected, i.e., no isolated groups. For studying opinion formation, it is important to have a connected network. In what follows, we use the parameters $H = 2$, $b = 2$, $\langle g \rangle = 10$, $L = 8$ in constructing the hierarchical network. The mean degree is taken to be $\langle k \rangle = 9$. A network property that has previously been found to be important in the opinion formation dynamics is the averaged length of shortest paths between any two nodes [13]. It is thus useful for the discussions below to show the length of shortest path ℓ as function of a in the networks to be studied (see Fig. 2). Several features should be noted. At large a, ℓ is longer as the network is characterized by groups of nodes connected by a few links of short social distance, and the network is kept connected for $H > 1$ for the reason that a node will belong to a group of different neighbors in different hierarchies. For large a, the shortest path is basically determined by the initial distribution of the nodes into the lowest level in each hierarchy, and thus ℓ does not change much for $a \geq 5$. For $0 < a < 4$, ℓ drops as a decreases. It is because a smaller positive value of a allows the establishment of links between nodes with a longer social distance in a hierarchy, and these links provide short-cuts to get from one node to another. The network has more links between groups of nodes. For $a < -1$, ℓ becomes small and saturated. For large b and $\langle g \rangle$, it is expected that the random network limit is reached at $a = -\ln b$. Note that the diameter of a random network is approximately $\log N / \log\langle k \rangle \sim 3.257$ and the shortest path of a random network is close to its diameter [28–30]. The saturated value at negative a is quite close to this estimated value and the discrepancy comes from the small values of b and $\langle g \rangle$ used. Note also that the difference in ℓ for large and small values of a is about 0.5, which is small. The length of shortest paths is more sensitive to other parameters, such as the number of levels L and mean group size $\langle g \rangle$, as these parameters control the number of nodes N in the system.

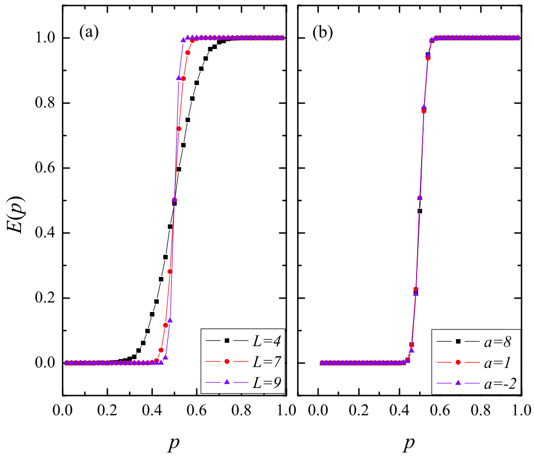

Fig. 3. (a) The exit probability $E(p)$ as a function of the initial density p of +1 opinion in three hierarchical networks with $L = 9$, 7, 4 in each hierarchy. The other parameters are $H = 2$, $b = 2$, $\langle g \rangle = 10$, $\langle k \rangle = 9$ and $\alpha = 1$. (b) The exit probability $E(p)$ as a function of p in three hierarchical networks with $L = 8$ and three different values of $a = 8$, 1, −2. The other parameters are the same as in (a). For each value of p, the data points represents an average over 40 different network configurations and for each configuration over 50 different realizations of initial opinion.

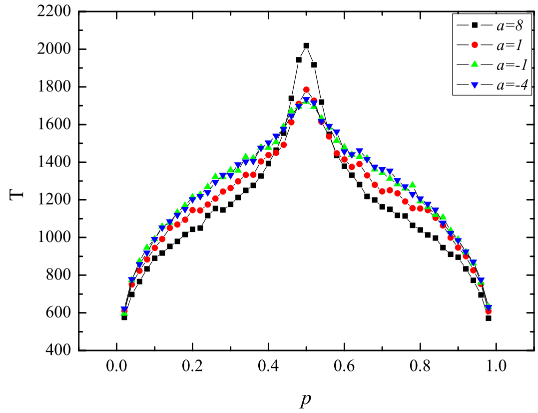

Fig. 4. The consensus time T as a function of the initial fraction of +1 opinion in a hierarchical network with $H = 2$, $L = 8$, $b = 2$, $\langle g \rangle = 10$ and $\langle k \rangle = 9$. Results for four different values of $a = 8$, 1 −1 −4 are shown. Each data point represents an average over 40 different configurations and for each configuration over 50 different initial opinion distributions.

3 Dynamics of opinion formation in social network

The fundamental quantities in studying opinion formation are the exit probability and the time to reach consensus [11–13]. The exit probability $E(p)$ is the probability that a system ends up with all the nodes taking on +1, given an initial fraction p of nodes taking on +1. Figure 3a shows $E(p)$ as a function of p for different values of L in hierarchical networks with $H = 2$, $b = 2$, $\langle g \rangle = 10$ and $a = 1$. Note that a larger value of L corresponds to a larger number of nodes N in the system. As L increases, $E(p)$ approaches a step function at $p = 0.5$. It is reasonable in that a larger system tends to suppress fluctuations, and the initial bias predetermines the final consensus. This behavior is similar to that observed in regular lattices [11]. In terms of the length of shortest paths, an estimate based on the random network limit gives $\ell \propto L$. As ℓ increases, the occurrence of a state that is different from the majority in the initial bias becomes increasingly improbable as nodes of minority opinion have to persuade groups of majority opinion many times.

Figure 3b shows the exit probability $E(p)$ as a function of the initial fractions p of +1 nodes for systems with three different values of a and a fixed number of nodes. The three values of a are chosen so that they correspond to different regions in the behavior of ℓ (see Fig. 2). It is noted that $E(p)$ behaves almost identically for different values of a, for systems with the same number of nodes N. As remarked in the discussion in Figure 2, varying a only changes the lengths of shortest paths by a small amount (∼0.5). Thus, $E(p)$ shows nearly no dependence on a, while changing L and thus N leads to more significant changes in $E(p)$.

While different values of a give similar $E(p)$, the time to reach consensus or the consensus time T is a quantity that is much more sensitive to the details of the network structure and hence the parameter a. The consensus time is expressed in units of Monte Carlo time step. In a Monte Carlo time step, each node on average has undergone an attempt on updating the state [13]. Figure 4 shows the mean consensus time T as a function of the initial fraction p of +1 opinion, for four different values of a. The mean consensus time corresponds to taking averages over different network configurations for the same network parameters *and* over different realizations of initial distributions of opinions for a given network configuration. The other parameters are chosen to be the same as those in Figure 2. Results of $a = 8$, 1, −1, and −4 are presented in Figure 4. From Figure 2, the length of shortest-paths drops as a decreases from $a = 8$ to $a \approx 0$, and ℓ for $a = -1$ and $a = -4$ are almost the same. From Figure 4, the consensus time $T(p)$ does depend on the parameter of homophyly a, even for systems with the same number of nodes. The change in $T(p)$ depends on the initial bias – an effect that is closely related to the network structure. In the vicinity of unbiased initial opinion $p \approx 0.5$, $T(p)$ becomes shorter as a (and thus ℓ) drops. For appreciable bias ($p > 0.6$ and $p < 0.4$), $T(p)$ becomes longer. For the two sets of results corresponding to $a = -1$ and $a = -4$, which have the same ℓ, $T(p)$ are nearly identical. These results indicate that the detail structure of the network as represented by ℓ and the initial bias are decisive factors for the dynamics of opinion formation. Previous results in networks constructed by randomly adding links to an underlying square lattice [13] show a similar behavior, but the feature is more apparent in the social networks studied here.

To examine the origin of the different behavior of $T(p)$ for the cases of unbiased and biased initial opinion, we

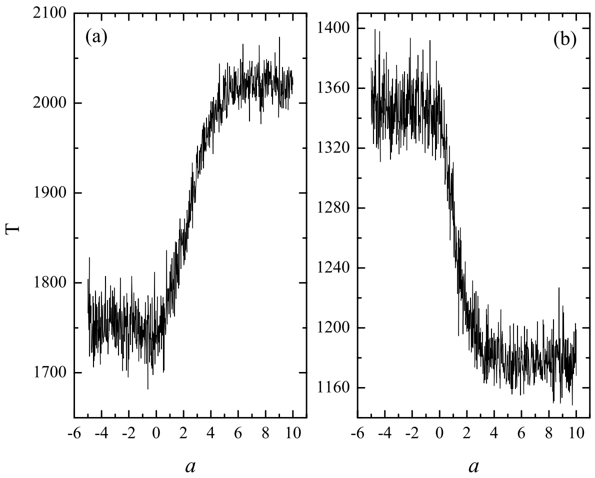

Fig. 5. The consensus time T as a function of the parameter of homophyly a for the case of (a) unbiased initial opinion $p = 0.5$, and (b) biased initial opinion $p = 0.3$, in a hierarchical network. The parameters in constructing the network are the same as that in Figure 2. Each data point is an average 40 different network configurations and for each configuration over 50 different realization of initial opinion distributions.

show $T(p)$ as a function of the parameter of homophyly a for $p = 0.5$ and $p = 0.3$ in Figure 5. While the results are noisy, the dependence of $T(p)$ on a is qualitatively different at these two p-values. We have checked that the behavior in Figure 5a is typical of $p \approx 0.5$, and that in Figure 5b is typical of $p > 0.6$ and $p < 0.4$. For unbiased initial opinion ($p = 0.5$), $T(p = 0.5)$ changes with a in the same way as the length of the shortest paths (compare Fig. 5a with Fig. 2). For biased initial opinion distribution ($p = 0.3$), T is longer for negative values of a and drops rapidly between $0 < a < 1$, and then rises slightly before saturating at large positive values of a. A clear indication that the network structure is important in opinion dynamics is that T changes most sensitively around $a \approx 0$ for different values of p. We note that $a = -\ln b = -0.693$ is where links between any pair of nodes are equally probable. However, the opposite trends of T versus a for $p = 0.5$ and $p = 0.3$ indicate that the initial bias is also a key ingredient in the dynamics of reaching a consensus.

The intricate interplay between network structure and initial bias as shown in Figures 5a and 5b can be understood qualitatively as follows. For $H > 1$, we have connected networks. For unbiased initial opinion ($p = 0.5$), in the early stage of negotiation and persuasion, groups negotiating for a local consensus have nearly as many nodes of $+1$ opinion as -1 opinion. As a result of the majority rule, clusters of $+1$ opinion and -1 opinion will be formed and the numbers of these clusters are about the same. For networks with $a < 0$, the structure is similar to a random network. This structure promotes a good mix of the nodes and thus has the advantageous of allowing these clusters of opposite opinions to meet readily. This provides a faster process in getting into a state of uniform opinion and results in a shorter T. As a becomes positive, the nodes tend to form groups with some links between the groups. With the change in network structure, it is more likely that negotiations first lead to local consensus for the groups, and then these groups try to influence each other. For $p = 0.5$, the local consensus within a group can either by $+1$ or -1 with equal probabilities. A global consensus, for which the consensus time refers to, can only be reached through the small number of links between the groups. As a increases, the nodes are connected in an increasingly localized fashion, resulting in a smaller chance for nodes in groups of different opinions to meet and thus a longer consensus time.

For biased initial opinion ($p = 0.3$), the situation is different. For networks with $a < 0$, the negotiation among the nodes gives a longer consensus time compared with the $a > 0$ case. It is because for small and positive a, the hierarchical structure starts to emerge. The groups will first arrive at a local opinion. Due to the initial bias, the local opinion in the groups is highly likely to be the initially biased opinion. That is to say, the local opinions are likely to be the same and the likelihood increases with the bias. If this is the case, there is no need for the nodes in different groups to meet and negotiate by making use of the links connecting nodes in different groups. For $a < 0$, the network is close to a random network. Although similar formation of local opinion of the same type also exists, the structure of the network has more extended links and thus a node may affect its neighbors in one step and its neighbors may be influenced by their own neighbors in another step. For $a > 0$, the links from a group to another are rare. This is the reason for the drop in the consensus time near $a = 0$. For strong initial bias, one would expect that the consensus time for positive a near $a \approx 0$ is determined by the time to reach a consensus within a group. For a given $p \neq 0$, there will be a chance, albeit small, of having groups of different opinions at intermediate steps and a consensus is finally reached by the links connecting the nodes in different clusters. For $a \approx 0$, such links readily exist and hence a consensus can be readily reached. For larger and positive a, the links connecting different groups only come from the multi-hierarchical structure of the network which depends on the initial distribution of the nodes among the lowest-level groups in each hierarchy. The probabilistic nature of the dynamics implies that it will take longer to pick the relevant nodes before a consensus can be reached. This is reflected in the slight increase in consensus time in the region of $a > 0$ in Figure 5b.

The Watts' network model [26] has the advantage that it interpolates the limits of finely divided groups of nodes with some links between them and random networks. The results in Figures 4 and 5 illustrate the effects of changing the network structure. The network structure gives different results for different initial biases. For finely divided groups with a strong initial bias, each group will reach the same local opinion and thus a global opinion is automatically reached, without using the links. For unbiased initial opinion, then the groups reach different local opinions and it takes much time for the links to act as mediators for reaching a global consensus.

4 Summary

We studied the dynamics of opinion formation in social hierarchical networks. The model gives a good representation of classifying or grouping of individuals in a population. The opinion formation mechanism is based on a local majority rule. The exit probability is found to change sensitively with the number of nodes in the system, but not with the parameter of homophyly. The consensus time, however, is found to be a result of non-trivial interplay between the detail of the network structure as characterized by the parameter of homophyly and the initial bias in opinion. For unbiased initial opinion, a common consensus is easier to be reached in a random network than a highly structured hierarchical network. For biased initial opinion, a common consensus is easier to be reached in a hierarchical network, as the local majority opinions of the groups may take on the biased opinion and hence be the same.

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References

1. R. Albert, A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002)
2. S.N. Dorogovtsev, J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002)
3. L. da F. Costa, F.A. Rodrigues, G. Travieso, P.R. Villas Boas, Adv. Phys. **56**, 167 (2007)
4. D.J. Watts, *Six Degrees: The Science of a Connected Age* (William Heinemann, London, 2003)
5. D.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998)
6. D. Vilone, C. Castellano, Phys. Rev. E **69**, 016109 (2004)
7. C. Castellano, D. Vilone, A. Vespignani, Europhys. Lett. **63**, 153 (2003)
8. S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.-U. Hwang, Phys. Rep. **424**, 175 (2006)
9. C. Castellano, S. Fortunato, V. Loreto, e-print `arXiv:0710.3256`
10. S. Galam, Physica A **274**, 132 (1999)
11. P.L. Krapivsky, S. Redner, Phys. Rev. Lett. **90**, 238701 (2003)
12. P. Chen, S. Redner, Phys. Rev. E **71**, 036101 (2005)
13. P.P. Li, D.F. Zheng, P.M. Hui, Phys. Rev. E **73**, 056128 (2006)
14. M.E.J. Newman, D.J. Watts, Phys. Rev. E **60**, 7332 (1999)
15. P.L. Krapivsky, Phys. Rev. A **45**, 1067 (1992)
16. L. Frachebourg, P.L. Krapivsky, Phys. Rev. E **53**, R3009 (1996)
17. I. Dornic, H. Chaté, J. Chavé, H. Hinrichsen, Phys. Rev. Lett. **87**, 045701 (2001)
18. K. Suchecki, V.M. Eguíluz, M. San Miguel, Phys. Rev. E **72**, 036132 (2005)
19. T.M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985)
20. J.M. Kleinberg, *Proc. 32nd ACM Symp. Theory Comput.* 163 (2000)
21. J.M. Kleinberg, Nature **406**, 845 (2000)
22. J. Travers, S. Milgram, Sociometry **32**, 425 (1969)
23. C. Korte, S. Milgram, J. Personal. Soc. Psychol. **15**, 101 (1970)
24. S. Milgram, Psychol. Today **2**, 60 (1967)
25. D.J. Watts, Annu. Rev. Sociol. 2004. **30**, 243 (2004)
26. D.J. Watts, P.S. Dodds, M.E.J. Newman, Science **296**, 1302 (2002)
27. D.F. Zheng, P.M. Hui, S. Trimper, B. Zheng, Physica A **352**, 659 (2005)
28. P. Erdös, A. Rényi, Publicationes Mathematicae **6**, 290 (1959)
29. P. Erdös, A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960)
30. L.L.F. Chung, Adv. Appl. Math **26**, 257 (2001)